Probability spaces, independence, and conditional probability: in principle

1. What are the two ingredients of a ***finite probability space***?

2. Joint probability distributions: suppose that $(X, \mathbb{P}_X)$ and $(Y, \mathbb{P}_Y)$ are finite probability spaces.
How do we define a probability distribution on the product $X \times Y$?
How do we interpret the meaning of this new probability space? Illustrate with an example.

3. Subspaces: suppose that B is an event in some probability space $(\Omega, \mathbb{P})$, and that $\mathbb{P}(B) > 0$.
   (a) How do the events of Ω give us a set of events for B?
   (b) How does the probability distribution $\mathbb{P}$ give us a probability distribution $\mathbb{P}_B$ on B?
   What must we be careful about when doing this, and why is it so important that B had $\mathbb{P}(B) > 0$?

4. Suppose that A and B are events in some probability space $(\Omega, \mathbb{P})$.
   (a) What does it mean for A and B to be ***mutually exclusive***? Illustrate with an example.
   (b) What does it mean for A and B to be ***independent***? Illustrate with an example.

5. Suppose that A and B are events in some probability space, and that $\mathbb{P}(B) > 0$.
   (a) How do we define the ***conditional probability*** $\mathbb{P}(A \mid B)$?
   (b) In words, what does this tell us?
   (c) How does this relate to our notion of *subspaces*?
   (d) What can be determined about this conditional probability if A and B are *independent*?

. . . and in practice

6. Let $(\Omega, \mathbb{P})$ be the probability space for two fair six-sided dice being rolled.
   (a) How can this probability space be considered as a product?
   (b) Compute the probabilities of each of the following, and determine which pairs of the events are independent:
      (i) The first roll is even.
      (ii) The first roll is a 1, 2, or 3.
      (iii) The second roll is a multiple of 3.
      (iv) The second roll is a 1, 2, or 3.
   (c) Let B be the event that the sum of the dice is between 3 and 6, inclusive.
      (i) Find the probability distribution for the the subspace given by B;
      what adjective describes this distribution, and why?
      (ii) Determine $\mathbb{P}(A_k \mid B)$ for each $k = 1, 2, 3, 4, 5, 6$, where A_k is the event that the first roll is k.
      Are any of these event A_k mututally exclusive of B? Are any independent of B?
      (iii) Determine $\mathbb{P}(A \mid B)$ and $\mathbb{P}(B \mid A)$, if A is the event that both rolls are equal.

7. Let $(\Omega, \mathbb{P})$ be the probability space for ten flips of a fair coin,
and let B be the event that exactly 5 of the flips are `H`.
   (a) Compute $P(A \mid B)$ and $P(B \mid A)$, if A is the event that the first five flips are all `H`.
   (b) Consider the subspace of $(\Omega, \mathbb{P})$ given by B,
   Compute the probabilities of the three events below in this subspace:
      (i) The first flip is `T`.
      (ii) The second flip is `T`.
      (iii) The first two flips are both `T`.
   Are the events (i) and (ii) independent events in this subspace? Discuss.
   (c) Again consider the subspace of $(\Omega, \mathbb{P})$ given by B.
   Compute the probabilities of the two events below and that of their intersection.
      (i) The first three flips are `H`.
      (ii) The last three flips are `H`.
   What describes the relationship between these two events in this subspace?